

Physics 137B, Spring 2004

Problem Set #1

Arman Cingoz

Problem 1

$$\text{Bohr radius } a_0 = \frac{4\pi\epsilon_0\hbar^2}{Zme^2}, E_1 = -\frac{\mu}{2\hbar^2} \left(\frac{Ze^2}{4\pi\epsilon_0} \right)^2$$

where $\mu = \frac{mM}{m+M}$, $m \leftarrow \text{mass of orbiting mass}$
 $M \leftarrow \text{mass of nucleus.}$

Ze = charge of nucleus

(a) A deuteron + electron

$$\downarrow \\ 1 \text{ proton} + 1 \text{ neutron} \Rightarrow Z=1$$

$$M = m_p + m_n \approx 2m_p \Rightarrow \mu = \frac{2me m_p}{me + 2m_p} \approx \frac{2me m_p}{2m_p} = me \quad (\text{since } m_p \gg me)$$

$$a_0 = 5.29 \times 10^{-11} \text{ m}$$

$$E_1 = \frac{-me}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 = -13.6 \text{ eV}$$

(b) He^+ singly ionized Helium $\rightarrow 2 \text{ protons} + 2 \text{ neutrons}, Z=2$

$$M \approx 4m_p$$

$$\therefore \mu = \frac{4me m_p}{me + 4m_p} \approx me \Rightarrow a_0 = 2.84 \times 10^{-11} \text{ m}$$

$$E_1 = -\frac{me}{2\hbar^2} \left(\frac{2e^2}{4\pi\epsilon_0} \right)^2 = 4 \left(-\frac{me}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right) = -54.4 \text{ eV}$$

(c) Positron $\rightarrow Z=1$

$$M=me \rightarrow \mu = \frac{me me}{me+me} = \frac{me^2}{2me} = \frac{me}{2}$$

$$\therefore a_0 = \frac{8\pi\epsilon_0\hbar^2}{mec^2}; E_1 = -\frac{me}{4\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 = \frac{1}{2} E_{1_H} = -6.8 \text{ eV}$$

$$= 1.05 \times 10^{-10} \text{ m}$$

(d) A proton + μ^- negative muon

$$\downarrow \\ Z=1$$

$$\mu = \frac{m_\mu M}{m_\mu + M} = \frac{m_\mu m_p}{m_\mu + m_p} = \frac{2.07(m_e)m_p}{2.07(m_e) + m_p} = 1.67 \times 10^{-28} \text{ kg}$$

$$a_\mu = \frac{4\pi\epsilon_0 h^2}{\mu e^2} = \frac{4\pi\epsilon_0 h^2}{m_e e^2} \frac{m_e}{\mu} = a_0 \left(\frac{m_e}{\mu} \right) = 2.84 \times 10^{-13} \text{ m}$$

$$E_1'' = -\frac{\mu}{2h^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 = -\frac{m_e}{2h^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \left(\frac{\mu}{m_e} \right) = E_1'' \left(\frac{\mu}{m_e} \right) = E_1'' (186) \\ = -2530 \text{ eV}$$

$$(e) V(r) = -\frac{GMm}{r} \quad \text{instead of} \quad V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} \quad \therefore \frac{Ze^2}{4\pi\epsilon_0} \rightarrow GMm = Gm_n^2$$

$$E_1 = -\frac{\mu}{2h^2} \left(\frac{Ze^2}{4\pi\epsilon_0} \right)^2 \rightarrow -\frac{\mu}{2h^2} (Gm_n^2)^2 = -\frac{m_n}{4h^2} G^2 m_n^4 = -\frac{m_n^5 G^2}{4h^2}$$

$$\left(\mu = \frac{m_n^2}{2m_n} = \frac{m_n}{2} \right)$$

$$\therefore E_1 = -\frac{(1.67 \times 10^{-27})^5 (6.67 \times 10^{-11})^2}{4 (1.05 \times 10^{-34})^2} = -1.31 \times 10^{-89} \text{ J} = -2.1 \times 10^{-68} \text{ eV}$$

$$a = \frac{4\pi\epsilon_0 h^2}{\mu e^2} \rightarrow \frac{h^2}{\mu GMm} = \frac{h^2}{\mu Gm_n^2} = \frac{2h^2}{Gm_n^3} = \frac{2 (1.05 \times 10^{-34})^2}{(6.67 \times 10^{-11}) (1.67 \times 10^{-27})^3}$$

$$= 0.07 \times 10^{24} \text{ m}$$

Problem 2

$$(a) E_r = -\frac{me}{2\hbar^2} \left(\frac{e^2}{4\pi\hbar\sigma} \right)^2$$

$E = T + V \Rightarrow T = E - V < 0$ in the forbidden region

$$-\frac{me}{2\hbar^2} \left(\frac{e^2}{4\pi\hbar\sigma} \right)^2 - -\frac{e^2}{4\pi\hbar\sigma r} < 0 \Rightarrow \frac{e^2}{4\pi\hbar\sigma r} < \frac{me}{2\hbar^2} \left(\frac{e^2}{4\pi\hbar\sigma} \right)^2$$

$$\Rightarrow r > \frac{e^2}{4\pi\hbar\sigma} / \left(\frac{me}{2\hbar^2} \left(\frac{e^2}{4\pi\hbar\sigma} \right)^2 \right) = \frac{2\hbar^2}{me} \left(\frac{4\pi\hbar\sigma}{e^2} \right) = 2a_0$$

$$(b) P_{\text{forbidden}} = \int_{2a_0}^{\infty} \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta |Y_{100}|^2 dr d\theta d\phi \leftarrow 2\pi$$

$$= \int_{2a_0}^{\infty} r^2 |R_{100}|^2 dr \underbrace{\int_0^{2\pi} \int_0^{\pi} \sin\theta |Y_{00}|^2 d\theta d\phi}_1$$

$$= \int_{2a_0}^{\infty} r^2 \left| \frac{2}{a_0^{3/2}} e^{-r/a_0} \right|^2 dr = \frac{4}{a_0^3} \int_{2a_0}^{\infty} r^2 e^{-2r/a_0} dr \quad \begin{matrix} \text{let } u = -\frac{2r}{a_0} \\ du = -\frac{2}{a_0} dr \end{matrix}$$

$$= \frac{4}{a_0^3} \left(-\frac{a_0}{2} \right) \int_{-4}^{-\infty} \left(\frac{a_0}{-2} \right)^2 u^2 e^u du = -\frac{1}{2} \int_{-4}^{-\infty} u^2 e^u du$$

$$= \frac{1}{2} \int_{-\infty}^{-4} x^2 e^x dx \quad \begin{matrix} \text{let } u = x^2 \\ du = 2x \\ v = e^x \end{matrix} = \frac{1}{2} \left[x^2 e^x \Big|_{-\infty}^{-4} - 2 \int_{-\infty}^{-4} x e^x dx \right]$$

$$= \frac{1}{2} \left[16e^{-4} - 2 \left(x e^x \Big|_{-\infty}^{-4} - \int_{-\infty}^{-4} e^x dx \right) \right]$$

$$= \frac{1}{2} \left[16e^{-4} - 2 \left(-4e^{-4} - e^x \Big|_{-\infty}^{-4} \right) \right] = \frac{1}{2} \left[16e^{-4} + 8e^{-4} + 2e^{-4} \right]$$

$$= \frac{1}{2} (26e^{-4}) = 18e^{-24} = 0.23$$

Problem 3

$$\Psi(\vec{r}, t=0) = \frac{1}{\sqrt{14}} [2\psi_{100}(\vec{r}) - 3\psi_{200}(\vec{r}) + \psi_{322}(\vec{r})]$$

(a) Recall the parity operator changes the coordinate system by
 $x \rightarrow -x, y \rightarrow -y, z \rightarrow -z$. In spherical coordinates, $(r, \theta, \phi) \rightarrow (r, \pi-\theta, \phi+\pi)$

$$P(\psi_{nlm}) = P(R_{nl}(r) Y_{lm}(\theta, \phi)) = \underbrace{P(R_{nl}(r))}_{R''_{nl}(r)} \underbrace{P(Y_{lm}(\theta, \phi))}_{Y''_{lm}(\pi-\theta, \phi+\pi)} = (-1)^l Y_{lm}(\theta, \phi)$$

$$\therefore P(\psi_{nlm}) = (-1)^l \psi_{nlm}$$

for eigenstates.

$$\therefore P(\psi_{100}) = \psi_{100}; P(\psi_{200}) = \psi_{200}; P(\psi_{322}) = (-1)^2 \psi_{322} = \psi_{322}$$

$$\therefore P(\Psi(\vec{r}, t=0)) = \Psi(\vec{r}, t=0)$$

So, this state is an eigenfunction of the parity operator (w/ eigenvalue +1).

$$(b) P_{\psi_{100}} = |\langle \psi_{100} | \Psi(\vec{r}, t=0) \rangle|^2 = \frac{4}{14} = \frac{2}{7}$$

$$P_{\psi_{200}} = |\langle \psi_{200} | \Psi(\vec{r}, t=0) \rangle|^2 = \frac{9}{14}$$

$$P_{\psi_{322}} = |\langle \psi_{322} | \Psi(\vec{r}, t=0) \rangle|^2 = \frac{1}{14}$$

$$P_{\text{any other state}} = 0.$$

$$(c) \langle \hat{E} \rangle = \langle \Psi | \hat{H} | \Psi \rangle = \frac{1}{14} (2\langle \psi_{100} \rangle - 3\langle \psi_{200} \rangle + \langle \psi_{322} \rangle) \hat{H} (2|\psi_{100}\rangle - 3|\psi_{200}\rangle + |\psi_{322}\rangle) \\ = \frac{1}{14} (2\langle \psi_{100} \rangle - 3\langle \psi_{200} \rangle + \langle \psi_{322} \rangle) (2E_1|\psi_{100}\rangle - 3E_2|\psi_{200}\rangle + E_3|\psi_{322}\rangle)$$

$$\frac{1}{14} \left(4E_1 \langle \psi_{100} | \psi_{100} \rangle + 9E_2 \langle \psi_{200} | \psi_{200} \rangle + E_3 \langle \psi_{322} | \psi_{322} \rangle \right)$$

(all cross terms = 0 since eigenstates are orthonormal)

$$= \boxed{\frac{1}{14} (4E_1 + 9E_2 + E_3)} = \frac{-me(\epsilon^2)}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{1}{14} \left[\frac{4}{\frac{1}{(n=1)^2}} + \frac{9}{\frac{1}{(n=2)^2}} + \frac{1}{\frac{1}{(n=3)^2}} \right] = -\frac{229}{504} \frac{me}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2$$

$\langle E \rangle = \sum_i |c_i|^2 E_i$ for $\Psi = \sum_i c_i \psi_i$

$$\langle L^2 \rangle = \langle \Psi | L^2 | \Psi \rangle = \frac{1}{14} (2 \langle \psi_{100} | - 3 \langle \psi_{200} | + \langle \psi_{322} |) L^2 (2 | \psi_{100} \rangle - 3 | \psi_{200} \rangle + | \psi_{322} \rangle)$$

$$= \frac{1}{14} (2 \langle \psi_{100} | - 3 \langle \psi_{200} | + \langle \psi_{322} |) (2 \hbar^2 (0)(0+1) | \psi_{100} \rangle - 3 \hbar^2 (0)(0+1) | \psi_{200} \rangle + \hbar^2 2(2+1) | \psi_{322} \rangle)$$

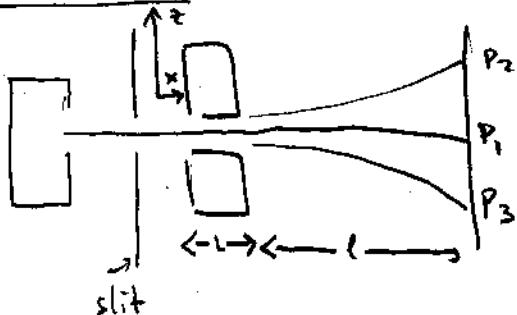
$$= \frac{1}{14} (2 \langle \psi_{100} | - 3 \langle \psi_{200} | + \langle \psi_{322} |) (6 \hbar^2 | \psi_{322} \rangle) = \boxed{\frac{6 \hbar^2}{14}}$$

$$\langle L_z \rangle = \langle \Psi | L_z | \Psi \rangle = \frac{1}{14} (2 \langle \psi_{100} | - 3 \langle \psi_{200} | + \langle \psi_{322} |) L_z (2 | \psi_{100} \rangle - 3 | \psi_{200} \rangle + | \psi_{322} \rangle)$$

$$= \frac{1}{14} (2 \langle \psi_{100} | - 3 \langle \psi_{200} | + \langle \psi_{322} |) (2 \hbar (0) | \psi_{100} \rangle - 3 \hbar (0) | \psi_{200} \rangle + 2 \hbar | \psi_{322} \rangle)$$

$$= \frac{1}{14} (2 \langle \psi_{100} | - 3 \langle \psi_{200} | + \langle \psi_{322} |) (2 \hbar | \psi_{322} \rangle) = \boxed{\frac{2 \hbar}{7}}$$

Problem 4



$$L = 0.1 \text{ m} \quad \frac{\partial B_z}{\partial z} = 10^3 \frac{\text{T}}{\text{m}}$$

$$l = 1 \text{ m}$$

$$T = 600 \text{ K}$$

$$\langle v \rangle = \left(\frac{3kT}{M} \right)^{1/2}$$

$$M = 1.78 \times 10^{-22} \text{ g} = 1.78 \times 10^{-25} \text{ kg}$$

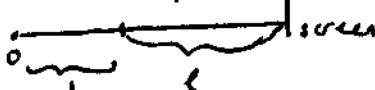
After the slit, we'll assume that the initial velocity is in the x-direction. Simple Newtonian mechanics:

$$F_z = M_{s_z} \frac{\partial B}{\partial z} = -g_s \mu_B \frac{S}{h} \frac{\partial B}{\partial z} = \pm \mu_B \frac{\partial B}{\partial z} = Ma_z$$

↑
2
acceleration.

$$\Rightarrow a_z = \pm \frac{\mu_B}{M} \frac{\partial B}{\partial z}$$

However, there is a force in the z-direction only within the gap b/w the magnets (to first order). We will use the left edge of the magnet as the origin of our coordinate system:



$$0 \leq x \leq L: x(t) = \langle v \rangle t \quad \therefore t = L/\langle v \rangle$$

$$z(t) = \frac{1}{2} a_z t^2 + z_0 \quad z(t) = \frac{1}{2} \frac{\mu_B}{M} \frac{\partial B}{\partial z} \left(\frac{L}{\langle v \rangle} \right)^2$$

$$v_z(t) = \int_0^t a_z dt = a_z \frac{L}{\langle v \rangle}$$

$$L \leq x \leq L+L: x(t) = \underset{\text{final}}{\langle v \rangle t} + x_0 \quad \therefore t = \frac{x - L - L}{\langle v \rangle} = \frac{L}{\langle v \rangle}$$

$$\text{No Force in z: } z(t_{\text{final}}) = v_z t + z_0 = a_z \frac{L}{\langle v \rangle} \left(\frac{L}{\langle v \rangle} \right) + \frac{1}{2} \frac{\mu_B}{M} \frac{\partial B}{\partial z} \left(\frac{L}{\langle v \rangle} \right)^2$$

However, notice that the distance between P_2 & P_3 is twice $z(t_{\text{final}})$ since I only calculated the distance b/w P_1 & P_2 .

$$P_2 - P_3 = 2 a_z \frac{L^2}{\langle v \rangle^2} + \frac{\mu_B}{M} \frac{\partial B}{\partial z} \left(\frac{L}{\langle v \rangle} \right)^2 = \frac{2 \mu_B}{M} \frac{\partial B}{\partial z} \frac{L^2}{\langle v \rangle^2} + \frac{\mu_B}{M} \frac{\partial B}{\partial z} \frac{L^2}{\langle v \rangle^2}$$

$$= \frac{\mu_B}{M} \frac{\partial B}{\partial z} \frac{L}{\langle v \rangle^2} [2L] = \frac{\mu_B}{M} \frac{\partial B}{\partial z} L (2L) \frac{M}{3kT} = \frac{\mu_B}{3kT} L (2L) \frac{\partial B}{\partial z}$$

$$= \frac{(9.27 \times 10^{-24} \text{ JT}^{-4})}{3(1.38 \times 10^{-23} \text{ J/K})} \cdot 10^3 \text{ T/m}^{-1} \cdot 0.1(2.1) = 0.078 \text{ m}$$

$$= 7.8 \text{ cm}$$

Problem 5

$$(a) \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \therefore \sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0-i & 1 \\ i & 0 \end{pmatrix} \therefore \sigma_y^2 = \begin{pmatrix} 0-i & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} 0-i & 1 \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \therefore \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(b) [S_x, S_y] = \frac{\hbar^2}{4} [\sigma_x, \sigma_y] = \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0-i & 1 \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0-i & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \frac{\hbar^2}{4} \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right] \\ = i\hbar \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar S_z \checkmark$$

$$[S_y, S_z] = \frac{\hbar^2}{4} [\sigma_y, \sigma_z] = \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \\ = i\hbar \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\hbar S_x \checkmark$$

$$[S_x, S_z] = \frac{\hbar^2}{4} [\sigma_x, \sigma_z] = \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \\ = i\hbar \frac{\hbar}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i\hbar \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -i\hbar S_y \checkmark$$

Since $[A, B] = -[B, A]$ we know that:

$$[S_y, S_x] = -i\hbar S_z, [S_z, S_y] = -i\hbar S_x, [S_z, S_x] = i\hbar S_y$$

$$\therefore [S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

$$(c) \sigma_x \sigma_y + \sigma_y \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0 \checkmark$$

$$(d) \text{Tr } \sigma_x = \text{Tr} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 \checkmark; \text{Tr } \sigma_y = \text{Tr} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 0 \checkmark; \text{Tr } \sigma_z = \text{Tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1 - 1 = 0 \checkmark$$

$$(e) \det \sigma_x = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 - 1 = -1 \checkmark$$

$$\det \sigma_y = \det \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 0 - (i)(-i) = 0 + (i)(i) = 0 - 1 = -1 \checkmark$$

$$\det \sigma_z = \det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1 - 0 = -1 \checkmark$$

(f) Eigenvalues of σ_z is obviously ± 1 since in this basis

σ_z is diagonal $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xrightarrow{\text{circles}} \lambda_1, \lambda_2$.

Let's diagonalize $\sigma_x \& \sigma_y$:

$$\underline{\sigma_x}: \det(\sigma_x - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \\ \therefore \lambda = \pm 1$$

$$\underline{\sigma_y}: \det(\sigma_y - \lambda I) = \det \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = \lambda^2 - (i)(-i) = \lambda^2 - 1 = 0 \\ \therefore \lambda = \pm 1$$

$$(g) S^2 = \vec{S} \cdot \vec{S} = S_x^2 + S_y^2 + S_z^2$$

$$\text{Since } \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}, \quad S_x^2 = S_y^2 = S_z^2 = \frac{1}{4}\mathbb{1}$$

$$\therefore S^2 = \frac{3}{4}\mathbb{1}$$

$$(h) \text{ Prove that } (\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B}) \quad B \neq J \text{ eqn 6.241.}$$

You may have noticed that this equation is a little sloppy. The left hand side is a matrix times a matrix = matrix. The second term on the right hand side is also a matrix. But the first term is a number. So the equation should be.

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = (\vec{A} \cdot \vec{B})\mathbb{1} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B}) \quad (\text{for } \vec{A} \& \vec{B} \text{ ordinary vectors})$$

I will solve the problem in two different ways. Since you were asked to use the explicit form of σ_i , I will use them to solve the problem in method 1.

$$\text{METHOD 1: } \vec{A} = (a_1, a_2, a_3) \quad \vec{B} = (b_1, b_2, b_3)$$

$$\vec{\sigma} \cdot \vec{A} = a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$$

$$\vec{\sigma} \cdot \vec{B} = b_1 \sigma_x + b_2 \sigma_y + b_3 \sigma_z = \begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$$

$$\therefore (\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_3 b_3 + (a_1 - ia_2)(b_1 + ib_2) & a_3(b_1 - ib_2) - b_3(a_1 - ia_2) \\ b_3(a_1 + ia_2) - a_3(b_1 + ib_2) & (a_1 + ia_2)(b_1 - ib_2) + a_3 b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_3 b_3 + a_1 b_1 + a_2 b_2 + i(a_1 b_2 - a_2 b_1) & i(a_2 b_3 - a_3 b_2) + a_3 b_1 - b_3 a_1 \\ i(a_2 b_3 - a_3 b_2) + a_1 b_3 - a_3 b_1 & a_3 b_3 + a_1 b_1 + a_2 b_2 + i(a_2 b_1 - b_2 a_1) \end{pmatrix}$$

Now I'll separate this into 2 matrices to get the desired result

$$= \underbrace{\begin{pmatrix} a_1 b_1 + a_2 b_2 + a_3 b_3 & 0 \\ 0 & a_1 b_1 + a_2 b_2 + a_3 b_3 \end{pmatrix}}_{(\vec{A} \cdot \vec{B}) \Downarrow} + i \underbrace{\begin{pmatrix} a_1 b_2 - a_2 b_1 & (a_2 b_3 - a_3 b_2) - i(a_3 b_1 - a_1 b_3) \\ (a_2 b_3 - a_3 b_2) + i(a_3 b_1 - a_1 b_3) & a_1 b_2 - a_2 b_1 \end{pmatrix}}_{\vec{\sigma} \cdot (\vec{A} \times \vec{B})}$$

$$\text{since } \vec{A} \times \vec{B} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - b_1 a_2 \end{pmatrix}$$

METHOD 2: There is a more elegant way that proves this identity without referring to the matrices representing σ_i .

The proof uses the fact in part (c) i.e $[\sigma_i, \sigma_j]_+ = 2\delta_{ij}$
 $([A, B]_+ \text{ is the anticomutator } AB + BA)$ & the fact that $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$
 (eqn 6.237 in the book. You can easily see this from the commutation relations of S_i $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k \Rightarrow S_i = \frac{\hbar}{2}\sigma_i$).

So, here is the derivation:

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \sum_i \sigma_i A_i \sum_j \sigma_j B_j = \sum_{ij} \sigma_i \sigma_j A_i B_j \quad \begin{matrix} (\text{components of } A \text{ & } B) \\ (\text{commute w/ } \sigma_i) \end{matrix}$$

$$\begin{aligned} \text{Now notice that } \sigma_i \sigma_j &= \frac{1}{2} (\sigma_i \sigma_j + \sigma_j \sigma_i) + \frac{1}{2} (\sigma_i \sigma_j - \sigma_j \sigma_i) \\ &= \frac{1}{2} [\sigma_i, \sigma_j]_+ + \frac{1}{2} [\sigma_i, \sigma_j] \\ &= \frac{1}{2} 2\delta_{ij} + \frac{1}{2} 2i\epsilon_{ijk}\sigma_k \end{aligned}$$

$$\begin{aligned} \therefore (\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) &= \sum_{ij} \sigma_i \sigma_j A_i B_j = \sum_{ij} (\delta_{ij} + i\sum_k \epsilon_{ijk}\sigma_k) A_i B_j \\ &= \underbrace{\sum_i A_i B_i}_{\vec{A} \cdot \vec{B}} + i \underbrace{\sum_{ijk} \epsilon_{ijk}\sigma_k A_i B_j}_{\underbrace{i \sum_k \sum_{ij} \epsilon_{ijk} A_i B_j}_{(A \times B)_k}} \\ &= \vec{A} \cdot \vec{B} + i \sigma_3 \cdot (\vec{A} \vec{B}) \end{aligned}$$